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Abstract

Let $\mathcal{F}$ and $\mathcal{G}$ be two collections of a total of n bivariate algebraic functions of constant maximum degree. The minimization diagrams of $\mathcal{F}$, $\mathcal{G}$ are the planar maps obtained by the xy-projections of the lower envelopes of $\mathcal{F}$, $\mathcal{G}$, respectively. We show that the combinatorial complexity of the overlay of the minimization diagrams of $\mathcal{F}$ and of $\mathcal{G}$ is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$. This result has several applications: (i) a near-quadratic upper bound on the complexity of the region in 3-space enclosed between the lower envelope of one such collection of functions and the upper envelope of another collection; (ii) an efficient and simple divide-and-conquer algorithm for constructing lower envelopes in three dimensions; and (iii) a near-quadratic upper bound on the complexity of the space of all plane transversals of a collection of simply-shaped convex sets in three dimensions.

1 Introduction

Let $\mathcal{F} = \{f_1, \ldots, f_n\}$ be a collection of n d-variate, possibly partially defined, functions, all algebraic of some constant maximum degree b (and if they are partially defined, the domain of definition of each f_i is also described by a constant number of polynomial equalities and inequalities of maximum degree b). Abusing the notation slightly, we will not distinguish between a function and its graph. The *lower envelope* $E_\mathcal{F}$ of $\mathcal{F}$ is defined as

$$E_\mathcal{F}(\mathbf{x}) = \min_i f_i(\mathbf{x}),$$

where the minimum is taken over all functions of $\mathcal{F}$ that are defined at $\mathbf{x}$. Similarly, we define the *upper envelope* $E'_\mathcal{F}$ of $\mathcal{F}$ as

$$E'_\mathcal{F}(\mathbf{x}) = \max_i f_i(\mathbf{x}).$$

The *minimization diagram* $M_\mathcal{F}$ of $\mathcal{F}$ is the decomposition of $\mathbb{R}^d$ into maximal connected relatively open cells, of any dimension, so that within each cell the same subset of functions appear on the envelope $E_\mathcal{F}$. (If the functions of $\mathcal{F}$ are partially defined, we also require that, over each cell c, each of the polynomials defining the domain of definition of any function that attains $E_\mathcal{F}$ over c has a fixed sign. Informally, this means that if a function f attains $E_\mathcal{F}$ over a cell c, then either c is fully contained in the boundary of the domain of f or is disjoint from that boundary.) The *combinatorial complexity* of $M_\mathcal{F}$ and of $E_\mathcal{F}$ is the number of cells of all dimensions in $M_\mathcal{F}$. The *maximization diagram* and its combinatorial complexity are defined in an analogous manner.

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Recently there has been significant progress in the analysis of the combinatorial complexity of lower envelopes of multivariate functions [10, 14]. In particular, it was shown in [14] that the maximum complexity of $M_{\mathcal{F}}$ is $O(n^{d+\varepsilon})$, for any $\varepsilon > 0$, where the constant of proportionality depends on ε, d and b. This result almost settles a major open problem and has already led to many applications [1, 10, 14]. In some applications, however, one has to consider the interaction between the lower envelope of one collection of functions and the upper envelope of another collection. A major application of this type, which has motivated the work on the present paper, is the analysis of the combinatorial complexity of the space of k-*transversals* of a collection $\mathcal{C}$ of n compact convex sets in d dimensions; a k-transversal is a k-flat that intersects all the sets of $\mathcal{C}$. Using an appropriate coordinate system for representing the space of k-flats in $\mathbb{R}^d$ (as is well known, the dimension of that space is $N = (k+1)(d-k)$), one can show that the space of k-transversals of $\mathcal{C}$ can be represented as the region enclosed between the upper envelope of one collection of functions and the lower envelope of another collection, where each function in the first (resp. second) collection represents all k-flats that are tangent to one of the given sets from below (resp. from above). Hence, the study of spaces of transversals calls for combinatorial (as well as algorithmic) analysis of the region enclosed between two envelopes in higher dimensions.

In this paper we provide such an analysis for the case of bivariate functions. We show that the combinatorial complexity of the region enclosed between the lower envelope of a collection of n low-degree bivariate algebraic functions and the upper envelope of another collection of n such functions, is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, where the constant of proportionality depends, as in the case of a single envelope, on ε and on the maximum degree of the given functions and, in case of partial functions, of their domain boundaries. In other words, the worst-case complexity of the region in question is asymptotically no worse than that of a single envelope.

The proof uses techniques that resemble those used in the proofs given in [10, 14], but requires several additional tricks. The basic result that we derive in this paper, which we consider to be interesting in its own right, is the analysis of the combinatorial complexity of the overlay of the minimization diagrams of the lower envelopes of two collections of bivariate functions. Notice that this problem is easy for the case of univariate functions, because the complexity of the overlay of the x-projections of two envelopes of univariate functions is proportional to the sum of the complexities of the individual envelopes. This is, however, not true for envelopes of bivariate functions; see Figure 1. Nevertheless, we show that the complexity of the overlay of the minimization diagrams of two collections of a total of n functions in 3-space is only $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$. This result not only implies the asserted bound on the complexity of the region enclosed between two envelopes, but also has several other useful applications, among which is a simple deterministic divide-and-conquer algorithm for computing lower envelopes, which we believe to be conceptually much simpler than the competing techniques of [3, 8, 14].

The paper is organized as follows. In Section 2 we prove the main result concerning the overlay of the projections of two envelopes in 3-space. In section 3 we apply the result to obtain:

- a near-quadratic upper bound on the complexity of the region enclosed between a lower envelope and an upper envelope;
- an efficient and simple divide-and-conquer algorithm for constructing lower envelopes in 3 dimensions; and
- a near-quadratic upper bound on the complexity of the space of all plane transversals of an arbitrary collection of simply-shaped convex sets in 3 dimensions.

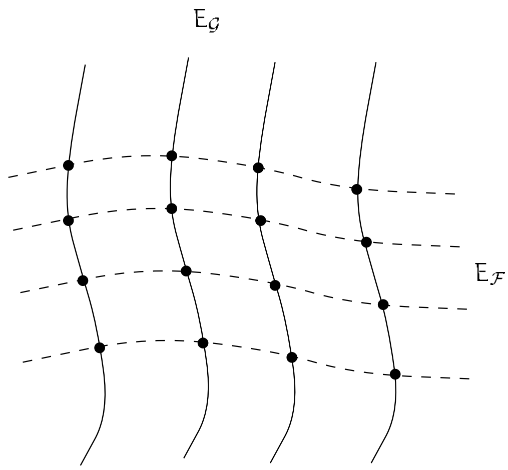

Figure 1: Overlay of two envelopes with quadratic complexity

2 Complexity of the Overlay of Two Envelopes

Let $\mathcal{F}$ and $\mathcal{G}$ be two given families of a total of n bivariate functions,[1] satisfying the following conditions:

(i) Each $f \in \mathcal{F} \cup \mathcal{G}$ is a continuous, totally or partially defined, bivariate algebraic function of constant maximum degree b; if f is only partially defined, the domain of definition of f is bounded by a constant number of algebraic arcs of constant maximum degree b.

(ii) The functions in $\mathcal{F} \cup \mathcal{G}$ are in *general position.* This excludes degenerate configurations where four function graphs meet at a point, a pair of graphs are tangent to each other, a singular point on one graph lies on an intersection curve between two other graphs, etc.)

Similar conditions were assumed in the papers [10, 14]. We refer the reader to these papers for more details, and for an argument that no real loss of generality is made by assuming general position. An appropriate variant of this argument shows that our main Theorem 2.4 below also holds for collections $\mathcal{F}$, $\mathcal{G}$ not in general position.

Our general position assumption implies that over each face of $M_{\mathcal{F}}$ the envelope is attained by a single function (or by no function at all), that over each edge the envelope is attained by two functions simultaneously or by the boundary of a single function graph, and that over each vertex of $M_{\mathcal{F}}$ the envelope is attained by three functions simultaneously, or by the intersection of the boundary of one function graph with another function, or by a point on the boundary of one function graph that lies directly below the boundary of another function graph, or below an intersection curve of two other functions (so that this higher point is vertically visible from the point on the lower boundary), or by a vertex of a function graph boundary (a point where two arcs forming this boundary meet). By the results of [10, 14], the combinatorial complexity (i.e., number of vertices, edges, and faces) of $M_{\mathcal{F}}$ and of $M_{\mathcal{G}}$ is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, where the constant of proportionality depends on ε and on the maximum degree b.

Let M denote the planar map obtained by superimposing $M_{\mathcal{F}}$ and $M_{\mathcal{G}}$. We refer to M as the *overlay* of $M_{\mathcal{F}}$ and of $M_{\mathcal{G}}$. For the sake of simplicity, we first prove an upper bound on M for the case when the functions in $\mathcal{F} \cup \mathcal{G}$ are totally defined.

[1] We will not distinguish between a function and its graph, and will use $\mathcal{F}$ (and $\mathcal{G}$) to denote a set of functions as well as the set of their graphs.

Theorem 2.1 *Let $\mathcal{F}$ and $\mathcal{G}$ be two collections of n totally defined bivariate functions, satisfying conditions (i) and (ii) stated above. Then the combinatorial complexity of the overlay of the minimization diagrams of $\mathcal{F}$ and $\mathcal{G}$, as defined above, is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$ (where the constant of proportionality depends on ε and on b).*

Proof: We use a two-stage counting argument to obtain a recurrence for the complexity of the overlay. We follow the notations introduced above, and note that the combinatorial complexity of M is proportional to the number of vertices of M. Indeed, by the general position assumption, the degree of each vertex v of M is either at most 3, if v is a vertex of $M_{\mathcal{F}}$ or of $M_{\mathcal{G}}$, or 4, if v is an intersection point of an edge of $M_{\mathcal{F}}$ with an edge of $M_{\mathcal{G}}$. We will refer to vertices of the second type as *crossings* in M. Clearly, it suffices to bound the number of crossings in M, because the number of its other vertices is $O(n^{2+\varepsilon})$.

For the purpose of analysis, we generalize the notion of a crossing, as follows. Let $\mathcal{A}(\mathcal{F})$ denote the arrangement of $\mathcal{F}$, namely the three-dimensional space decomposition induced by the graphs of the functions of $\mathcal{F}$ (see [9] for a more detailed definition). The *level* of a point w in $\mathcal{A}(\mathcal{F})$ is defined as the number of surfaces of $\mathcal{F}$ that lie vertically below w (note that 0-level points are precisely those that lie on or below the lower envelope $E_{\mathcal{F}}$). Let e be an edge of $\mathcal{A}(\mathcal{F})$. Clearly, the level of all points on e is the same, so we define the level of e to be the level of any point on e. We define the arrangement $\mathcal{A}(\mathcal{G})$, and the level of a point or of an edge in this arrangement, in an analogous manner for the collection $\mathcal{G}$. Let e be an edge of $\mathcal{A}(\mathcal{F})$, and let e' be an edge of $\mathcal{A}(\mathcal{G})$, such that the xy-projections of e and e' cross each other at a point σ. Let ξ, ξ' be the levels of the respective edges e, e'. Then we say that (e, e', σ) is an *edge-crossing* in $(\mathcal{A}(\mathcal{F}), \mathcal{A}(\mathcal{G}))$ at *level* (ξ, ξ'). If the point σ is not important, or is clear from the context, we just use (e, e') to denote the edge crossing (e, e', σ) (by our assumptions, for any pair of edges (e, e'), as above, there is only a constant number of points σ that appear in edge-crossings of the form (e, e', σ)). Note that the original crossings in M correspond to edge-crossings at level $(0, 0)$. (We slightly confuse the notation here, because crossings in M involve arcs in the xy-plane, whereas edge-crossings in $(\mathcal{A}(\mathcal{F}), \mathcal{A}(\mathcal{G}))$ involve arcs (of these arrangements) in 3-space; this abuse will also take place in what follows.) Let $C_{p,q}(\mathcal{F}, \mathcal{G})$ denote the number of edge-crossings in $(\mathcal{A}(\mathcal{F}), \mathcal{A}(\mathcal{G}))$ whose level is (p', q') for some $p' \le p$, $q' \le q$, and let

$$C_{p,q}(n) = \max C_{p,q}(\mathcal{F}, \mathcal{G}),$$

where the maximum is taken over all collections $\mathcal{F}$ and $\mathcal{G}$, as above, such that $|\mathcal{F}| + |\mathcal{G}| = n$. The goal is thus to obtain a sharp upper bound for $C_{0,0}(n)$.

Let k be a threshold parameter, whose value will be specified later on. Let e be an edge of $M_{\mathcal{F}}$, and let V_e be the vertical 2-manifold obtained as the union of all z-vertical lines passing through points of e. The intersection of the graph of each function $g \in \mathcal{G}$ with V_e is an algebraic arc of constant maximum degree, so each pair of these arcs intersect in at most some constant number, s, of points (where s depends only on the maximum degree of the functions of $\mathcal{F} \cup \mathcal{G}$ and of their graph boundaries, but not on e). Let $\mathcal{A}^{(e)}(\mathcal{G})$ denote the cross-section of $\mathcal{A}(\mathcal{G})$ with V_e, and let $C_{0,q}(e, \mathcal{G})$ denote the number of edge-crossings of the form (e, e') whose level is $(0, q')$, for any $q' \le q$. See Figure 2 for an illustration. A simple but crucial observation is:

Lemma 2.2 *Let e' be an edge of $\mathcal{A}(\mathcal{G})$. Then (e, e', σ) is an edge-crossing in $(\mathcal{A}(\mathcal{F}), \mathcal{A}(\mathcal{G}))$ at level $(0, \xi)$, if and only if the point of $e' \cap V_e$ that lies on the z-vertical line through σ is a vertex at level ξ in $\mathcal{A}^{(e)}(\mathcal{G})$, and vice-versa.*

This lemma implies that each crossing of e in M corresponds to a vertex in the cross-section $E^{(e)}_{\mathcal{G}}$ of the lower envelope $E_{\mathcal{G}}$ within V_e. Let $\mathcal{G}^{(e)} \subseteq \mathcal{G}$ be the subset of functions of $\mathcal{G}$ that appear on $E^{(e)}_{\mathcal{G}}$, and let $t = |\mathcal{G}^{(e)}|$. By the standard Davenport-Schinzel theory [2, 11], $C_{0,0}(e, \mathcal{G}) \le \lambda_s(t)$, where a is an appropriate constant (depending on the maximum degree of the surfaces in $\mathcal{F} \cup \mathcal{G}$), and where $\lambda_s(t)$ is the maximum length of a (t, s)-Davenport-Schinzel sequence.

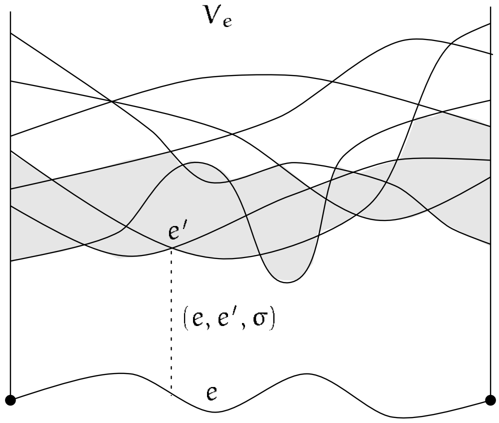

Figure 2: The arrangement $\mathcal{A}^{(e)}(\mathcal{G})$; the shaded region consist of points at level ≤ 3.

If $t \leq k$ then, by Lemma 2.2, there are at most $\lambda_s(k)$ edge-crossings at level $(0,0)$ involving e. Since the number of edges in $M_{\mathcal{F}}$ is $O(n^{2+\varepsilon})$, the overall number of crossings involving such edges is at most $O(\lambda_s(k)n^{2+\varepsilon})$. Thus, assume that $t > k$.

Let g, g' be a pair of distinct functions in $\mathcal{G}^{(e)}$. By continuity, g and g' must intersect within V_e at least once. Thus each function $g \in \mathcal{G}^{(e)}$ must cross at least $t-1$ other functions of $\mathcal{G}$ within V_e, that is, each function $g \in \mathcal{G}^{(e)}$ is incident to at least $t-1$ vertices of $\mathcal{A}^{(e)}(\mathcal{G})$. Since the graph of g contains points at level 0 in this cross section, it follows that g is incident to at least k vertices of $\mathcal{A}^{(e)}(\mathcal{G})$ at level $\leq k$. The number of vertices of $\mathcal{A}^{(e)}(\mathcal{G})$ at level $\leq k$ is therefore $\Omega(tk)$, which, by Lemma 2.2, implies

$$C_{0,k}(e,\mathcal{G}) = \Omega(tk) = \Omega\left(k\frac{t}{\lambda_s(t)} \cdot C_{0,0}(e,\mathcal{G})\right) \geq \frac{k}{\beta(n)} \cdot C_{0,0}(e,\mathcal{G}), \tag{1}$$

where $\beta(n) = \Theta(\lambda_s(n)/n)$ is an extremely slowly growing function of n [2, 11].

Summing (1) over all edges of $M_{\mathcal{F}}$ that cross more than k edges of $M_{\mathcal{G}}$, adding the bound for the other edges of $M_{\mathcal{F}}$, and observing that each edge crossing in $C_{0,k}(\mathcal{F},\mathcal{G})$ is counted in this manner exactly once, we obtain:

$$C_{0,0}(\mathcal{F},\mathcal{G}) = \sum_{e \in M_{\mathcal{F}}} C_{0,0}(e,\mathcal{G}) \leq \frac{\beta(n)}{k} C_{0,k}(\mathcal{F},\mathcal{G}) + O(kn^{2+\varepsilon}),$$

which implies:

$$C_{0,0}(n) \leq \frac{\beta(n)}{k} C_{0,k}(n) + O(kn^{2+\varepsilon}). \tag{2}$$

We next bound $C_{0,k}(n)$ in terms of $C_{k,k}(n)$. Let e' be an edge of $\mathcal{A}(\mathcal{G})$ at some level $\xi' \leq k$, let $V_{e'}$ be the vertical 2-manifold erected from e', defined as above, and consider the cross section $\mathcal{A}^{(e')}(\mathcal{F})$ of $\mathcal{A}(\mathcal{F})$ within $V_{e'}$. Let t denote the number of functions of $\mathcal{F}$ that appear on the lower envelope of $\mathcal{A}^{(e')}_{\mathcal{F}}$. If $t \leq k$, then e' contributes at most $\lambda_s(k)$ edge crossings to $C_{0,k}(\mathcal{F}, G)$. Since there are only $O(k^{1-\varepsilon}n^{2+\varepsilon})$ edges of $\mathcal{A}(\mathcal{G})$ at level at most k (see, e.g., [13, 15]), the number of crossings as above is $O(\lambda_s(k)k^{1-\varepsilon}n^{2+\varepsilon}) = O(k^2n^{2+\varepsilon})$, for any $\varepsilon > 0$. We can thus assume that $t > k$.

We can now repeat, within $V_{e'}$, the preceding analysis, replacing $\mathcal{G}$ by $\mathcal{F}$, so as to conclude that the number of edge crossings of the form (e, e') at level (ξ, ξ'), for all $\xi \leq k$, is $\Omega(tk)$. Following the same arguments as above, and noting that each such crossing (e, e') is counted in this manner at most once, we easily obtain the recurrence:

$$C_{0,k}(n) \le \frac{\beta(n)}{k} C_{k,k}(n) + O(k^2 n^{2+\varepsilon}) . \quad (3)$$

Next, we estimate $C_{k,k}(n)$, by using the probabilistic technique of Clarkson and Shor [7] (see also [13]). Since each edge crossing (e, e') is defined by four functions (two functions $f_1, f_2 \in \mathcal{F}$ whose intersection curve contains e, and two functions $g_1, g_2 \in \mathcal{G}$ whose intersection curve contains e'), the Clarkson-Shor technique is easily seen to yield the following inequality:

$$C_{k,k}(n) = O(k^4) \cdot C_{0,0}\left(\frac{n}{k}\right) . \quad (4)$$

Combining (2), (3), and (4), we thus obtain

$$\begin{aligned} C_{0,0}(n) &= O((k + k\beta(n))n^{2+\varepsilon}) + \frac{\beta^2(n)}{k^2} \cdot O(k^4) \cdot C_{0,0}\left(\frac{n}{k}\right) & (5) \\ &= O(k\beta(n)n^{2+\varepsilon}) + O(k^2\beta^2(n)) \cdot C_{0,0}\left(\frac{n}{k}\right) . & (6) \end{aligned}$$

The solution of this recurrence is easily seen to be $O(n^{2+\delta})$, for any $\delta > \varepsilon$. This is shown by induction, choosing $k = \beta^{1+2/\delta}(n)$ and using the fact that $\beta(n)$ is an extremely slowly growing function of n. This concludes the proof of Theorem 2.4. □

If we try to extend the above proof to partially defined functions, two difficulties arise. First, the level of an edge in $\mathcal{A}(\mathcal{F})$ (or $\mathcal{A}(\mathcal{G})$) is no longer well defined, because the level of all points on an edge is not necessarily the same. Second, the appearance of two functions g, g' on the lower envelope of $\mathcal{G}^{(e)}$ does not imply that g and g' intersect inside V_e (and similarly for $\mathcal{F}^{(e')}$). We circumvent these problems as follows. We split each edge $e \in \mathcal{A}(\mathcal{F})$ at a point $p \in e$ if the vertical ray emanating from p in the negative z-direction intersects the boundary of (the graph of) some function $f \in \mathcal{F}$. We split the edges of $\mathcal{A}(\mathcal{G})$ in a similar manner. Notice that the edges of $E_{\mathcal{F}}$ (resp. of $E_{\mathcal{G}}$) are not subdivided, becuase there is no function graph of $\mathcal{F}$ (resp. $\mathcal{G}$) lying below these edges. The level of all points on a newly-formed edge, in either arrangement, is now constant, so the level of any edge, and therefore also of any edge-crossing, is well defined. We call an edge-crossing (e, e') a *boundary edge-crossing* if e or e' is contained in the boundary of some function graph in $\mathcal{F}$ or in $\mathcal{G}$, respectively.

Lemma 2.3 *The overall number of boundary edge-crossings (e, γ), where e is at level 0 and γ is a boundary edge in $\mathcal{A}(\mathcal{G})$, is $O(n\lambda_{s'}(n))$, where s' is an appropriate constant depending on the maximum degree of the functions in $\mathcal{F} \cup \mathcal{G}$.*

Proof: Let γ be a boundary arc of a function graph in $\mathcal{G}$ (there are a total of $O(n)$ such arcs), and let V_γ denote the vertical 2-manifold erected from γ, defined exactly as V_e in the proof of Theorem 2.1. Let $E_{\mathcal{F}}^{(\gamma)}$ denote the cross section $E_{\mathcal{F}} \cap V_\gamma$. As in Lemma 2.2, each boundary edge-crossing involving γ in the present lemma corresponds to a breakpoint of $E_{\mathcal{F}}^{(\gamma)}$, and, by the standard Davenport-Schinzel theory, the number of such breakpoints is $O(\lambda_{s'}(n))$, for an appropriate constant s'. This is easily seen to imply the lemma. □

We now establish the recurrence (2) for partially defined functions, in the same way as above, but with the following additional modifications. Let k be a threshold parameter. We split each edge e of $E_{\mathcal{F}}$ at a point σ if there exist an integer $l \le k$ and a boundary edge $e' \in \mathcal{A}(\mathcal{G})$ such that (e, e', σ^*) is a boundary edge-crossing at level $(0, l)$, where σ^* is the xy-projection of σ. This step ensures that there is no boundary edge-crossing at level $(0, l)$ for any $l \le k$. By Lemma 2.3, we introduce a total of at most $O(n\lambda_{s'}(n))$ new vertices, over all edges e of $E_{\mathcal{F}}$, so the number of edges in $M_{\mathcal{F}}$ is still $O(n^{2+\varepsilon})$.

Fix an edge e of the (refined) lower envelope $E_{\mathcal{F}}$. Define $V_e, \mathcal{A}^{(e)}(\mathcal{G})$, and $E_{\mathcal{G}}^{(e)}$ as above. By construction, the level of any point on $\partial g \cap V_e$, for any $g \in \mathcal{G}$, is greater than k. Let $\mathcal{G}^{(e)}$ be the set of connected components of $g \cap V_e$, for $g \in \mathcal{G}$, that appear on $E_{\mathcal{G}}^{(e)}$, and let $t = |\mathcal{G}^{(e)}|$. The case $t \leq k$ is handled exactly as in the preceding proof, so assume that $t > k$. We claim that any arc $\gamma \in \mathcal{G}^{(e)}$ is incident to at least k vertices of $\mathcal{A}^{(e)}(\mathcal{G})$ whose levels are $\leq k$. There are two cases to consider:

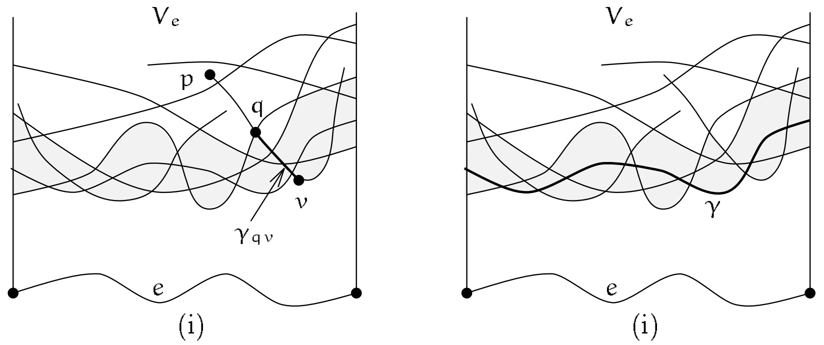

Figure 3: Illustration of the claim (for $k = 3$): (i) γ has a point p at level $\geq k$; (ii) the level of all points on γ is $< k$.

(i) γ has a point p whose level is $\geq k$. Let ν be a point on $\gamma \cap E_{\mathcal{G}}^{(e)}$; without loss of generality, assume that ν lies to the right of p. Let q be the rightmost point on γ to the left of ν whose level is k, and let $\gamma_{q\nu}$ denote the portion of γ between q and ν. The level of all points on $\gamma_{q\nu}$ is at most k. Let $g' \in \mathcal{G}$ be any of the k function graphs lying below q. Obviously, g' cannot lie below ν. Moreover, by construction, no point of $\partial g'$ can lie below $\gamma_{q\nu}$, so g' has to intersect $\gamma_{q\nu}$ (see Figure 3(i)). Since there are k function graphs of $\mathcal{G}$ lying below q, the arc $\gamma_{q\nu}$ contains at least k vertices of $\mathcal{A}^{(e)}(\mathcal{G})$, and the level of each of them is at most k.

(ii) The level of all points on γ is $< k$. In this case, by construction, the endpoints of γ must lie on the vertical boundary edges of V_e. Let γ' be another arc in $\mathcal{G}^{(e)}$. If the endpoints of γ' also lie on the vertical boundary edges of V_e, then, as argued in the proof of Theorem 2.1, γ and γ' intersect within V_e. Otherwise, γ' has an endpoint p that lies inside V_e. By construction, the level of p is $> k$. Since the level of all points in γ is $\leq k$, the endpoint p lies above γ, implying. again, that γ and γ' intersect (see Figure 3 (ii)). Since $t > k$, we obtain at least k such intersections with γ, and the level of each of these intersection points is $\leq k$.

This completes the proof of the claim. Hence $\mathcal{A}^{(e)}(\mathcal{G})$ has $\Omega(tk)$ vertices at level $\leq k$. Following the same argument as in the proof of Theorem 2.1, we can obtain the recurrence (2) in this case too.

A similar argument can be applied to obtain the recurrence (3) for the case of partial functions. The only difference is that we now split an edge $e' \in \mathcal{A}(\mathcal{G})$ at a point p if there are integers $j, j' \leq k$ and a boundary edge e of some function in $\mathcal{F}$ such that (e, e', p^*) is an edge-crossing at level (j, j'), where p^* is the xy-projection of p. Using the proof of Lemma 2.3, in conjunction with the Clarkson-Shor technique, we can show that the number of newly added vertices in $\mathcal{A}(\mathcal{G})$ is $O(k^2 n \lambda_{s'}(n/k))$, for an appropriate constant s'. The recurrence (3) now following by the same analysis as above. This is turn yields the same final recurrence (5) for $C_{0,0}(n)$, whose solution, as above, is $O(n^{2+\varepsilon})$. We thus obtain the main result of the paper:

Theorem 2.4 *Let $\mathcal{F}$ and $\mathcal{G}$ be two collections of n, possibly partially defined, bivariate functions, satisfying the conditions stated above. Then the combinatorial complexity of the overlay of the minimization diagrams of $\mathcal{F}$ and $\mathcal{G}$, as defined above, is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$ (where the constant of proportionality depends on ε and on the maximum degree b).*

3 Applications

In this section we present some applications of Theorem 2.4.

3.1 Computing lower envelopes in 3-space

Let $\mathcal{F}$ be a collection of n bivariate functions satisfying the above conditions. Our goal is to construct the lower envelope $E_{\mathcal{F}}$ of $\mathcal{F}$. This is equivalent to constructing the minimization diagram $M_{\mathcal{F}}$, as defined above, so that each face ϕ of $M_{\mathcal{F}}$ is labeled with the unique function of $\mathcal{F}$ (if exists) attaining $E_{\mathcal{F}}$ over ϕ. Several algorithms for this construction have recently been designed (see [3, 8, 14]), but they are either rather complicated or require the use of randomization. Here we present a simple deterministic algorithm based on the divide-and-conquer approach.

The algorithm partitions $\mathcal{F}$ into two subcollections, $\mathcal{F}_1$, $\mathcal{F}_2$, of roughly $n/2$ functions each, constructs recursively the minimization diagrams $M_{\mathcal{F}_1}$, $M_{\mathcal{F}_2}$, and then merges these diagrams to obtain the final minimization diagram $M_{\mathcal{F}}$.

The merging step is done as follows. We first compute the superposition M of $M_{\mathcal{F}_1}$ and $M_{\mathcal{F}_2}$. This can be done, e.g., by applying a standard sweep-line procedure, whose running time is $O((|M_{\mathcal{F}_1}|+|M_{\mathcal{F}_2}|+|M|)\log n)$; by Theorem 2.4, this is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$. We can implement the sweep so that it also constructs the *vertical decomposition* of M; this is a refinement of M, obtained by drawing a vertical segment upwards and downwards (in the y-direction) from each vertex of M and from each point on any edge of M that has y-vertical tangency, and by extending each segment until it hits another edge of M. The number of resulting cells, usually referred to as 'pseudo-trapezoids', is proportional to the complexity of M, namely it is also $O(n^{2+\varepsilon})$.

Let c be a pseudo-trapezoid in this vertical decomposition. Note that, over c, the envelope $E_{\mathcal{F}_1}$ is attained by a single function $f_1 \in \mathcal{F}_1$ (or by no function at all), and $E_{\mathcal{F}_2}$ is attained by a single function $f_2 \in \mathcal{F}_2$ (or by no function at all). Hence, the envelope $E_{\mathcal{F}}$ is equal over c to $\min\{f_1, f_2\}$ if both functions exists, or to one of these functions if the other does not exist, or, if both functions do not exist, $E_{\mathcal{F}}$ is undefined over c. In any case, we can compute $E_{\mathcal{F}}$ over c in constant time.[2] We repeat this computation over all pseudo-trapezoids of M, in overall $O(n^{2+\varepsilon})$ time, and thus obtain the entire envelope $E_{\mathcal{F}}$. We still need to apply a final clean-up stage, in which the computed portions of $E_{\mathcal{F}}$ are properly glued together, removing, as appropriate, any redundant data concerning the behavior of $E_{\mathcal{F}}$ over edges of the pseudo-trapezoids of M. This stage also produces the final minimization diagram $M_{\mathcal{F}}$, with its faces labeled in the required manner. We omit the routine details of this step, and note that it also takes only $O(n^{2+\varepsilon})$ time. It follows that the cost of the entire divide-and-conquer process is also $O(n^{2+\varepsilon})$. In conclusion, we thus have:

Theorem 3.1 *The lower envelope of a collection of n bivariate functions, as above, can be computed, in an appropriate model of computation, by a deterministic divide-and-conquer algorithm, in time $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, where the constant of proportionality depends on ε and on the maximum algebraic degree of the given functions (and of their domain boundaries).*

3.2 Complexity of the region enclosed between two envelopes in 3-space

Let $\mathcal{T}$ and $\mathcal{B}$ be two given families of a total of n, possibly partially-defined, bivariate functions satisfying the conditions stated in Section 2. We denote by $\mathcal{L}_{\mathcal{T}}$ the lower envelope of the 'top' family $\mathcal{T}$, and by $\mathcal{U}_{\mathcal{B}}$ the upper envelope of the 'bottom' family $\mathcal{B}$. We consider the region $K =$

[2] We are implicitly assuming an appropriate model of computation, in which computing the pointwise minimum of two given functions, as well as various primitive operations involving edges of the minimization diagrams, can be performed in constant time. For example, we can use precise rational arithmetic to perform each of these operations in constant time, using standard techniques from computational real algebraic geometry; see, e.g., [12].

$\{(x, y, z) \mid \mathcal{U}_{\mathcal{B}}(x, y) \le z \le \mathcal{L}_{\mathcal{T}}(x, y)\}$ of points lying between the two envelopes, and our goal is to derive an $O(n^{2+\varepsilon})$ bound on the combinatorial complexity of K.

We establish this bound as follows. Let $M_{\mathcal{T}}$, $M_{\mathcal{B}}$ denote the minimization and maximization diagrams of the envelopes $\mathcal{L}_{\mathcal{T}}$, $\mathcal{U}_{\mathcal{B}}$, respectively. By Theorem 2.4, the combinatorial complexity of the overlay M of these two planar maps is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$. Construct the vertical decomposition of M, as defined above. As noted, the number of pseudo-trapezoids of this decomposition is proportional to the complexity of M, i.e., it is $O(n^{2+\varepsilon})$. Observe that, for each resulting pseudo-trapezoid τ, there is a single function $f \in \mathcal{T}$ and a single function $g \in \mathcal{B}$ such that $\mathcal{L}_{\mathcal{T}} \equiv f$ and $\mathcal{U}_{\mathcal{B}} \equiv g$ over τ (if the given functions are only partially defined then either f or g or both may not exist at all, in which case the corresponding envelope(s) are undefined over τ). This implies that the portion of K that projects into τ has constant complexity—it is defined by the interaction between f, g, and the functions defining the (at most 4) edges of τ. Since the number of pseudo-trapezoids is $O(n^{2+\varepsilon})$, we immediately obtain:

Theorem 3.2 *The combinatorial complexity of the region enclosed between a lower envelope and an upper envelope of two respective collections of n bivariate functions satisfying the above conditions, is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$, where the constant of proportionality depends on ε and on the maximum algebraic degree of the given functions (and of their domain boundaries).*

It is also easy to construct the desired region K, in a manner that resembles the divide-and-conquer algorithm presented above. That is, we compute $\mathcal{L}_{\mathcal{T}}$ and $\mathcal{U}_{\mathcal{B}}$ separately, in time $O(n^{2+\varepsilon})$, using the algorithm of the preceding subsection. Next we compute the overlay of the minimization diagram $M_{\mathcal{T}}$ and of the maximization diagram $M_{\mathcal{B}}$, using the same sweep technique described above, and decompose the resulting map into pseudo-trapezoids. Finally we compute the portions of K over each pseudo-trapezoid separately, and 'glue' together the resulting pieces to obtain the whole K. It is easily verified that the overall complexity of the algorithm is $O(n^{2+\varepsilon})$, so we have:

Theorem 3.3 *The region enclosed between two envelopes in 3-space, as above, can be computed in (deterministic) time $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$.*

3.3 Complexity of the space of plane transversals

In this subsection we obtain new bounds on the combinatorial complexity of the space of plane transversals of a collection of simply-shaped convex sets in 3-space. Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a collection of n compact convex sets in 3-space. A plane π is a *transversal* of $\mathcal{C}$ if it intersects every set in $\mathcal{C}$. The space of all plane transversals of $\mathcal{C}$ is denoted by $T(\mathcal{C})$.

It is more convenient to represent $T(\mathcal{C})$ in the dual space, where each nonvertical plane $z = \xi x + \eta y + \zeta$ is mapped to a point (ξ, η, ζ), and each point (u, v, w) is mapped to a plane $z = -ux - vy + w$. Note that a plane $z = \xi x + \eta y + \zeta$ intersects a compact convex set C if and only if $\phi_C(\xi, \eta) \le \zeta \le \psi_C(\xi, \eta)$, where $\phi_C(\xi, \eta)$, $\psi_C(\xi, \eta)$ are defined so that the plane $z = \xi x + \eta y + \phi_C(\xi, \eta)$ (resp. $z = \xi x + \eta y + \psi_C(\xi, \eta)$) is tangent to C from below (resp. from above). Thus, in the dual space, the set of all plane transversals of $\mathcal{C}$ is the set

$$\left\{(\xi, \eta, \zeta) \mid \max_{C \in \mathcal{C}} \phi_C(\xi, \eta) \le \zeta \le \min_{C \in \mathcal{C}} \psi_C(\xi, \eta)\right\} .$$

That is, $T(\mathcal{C})$ is, in the dual space, the region enclosed between a lower envelope and an upper envelope of two respective collections of functions.

We can therefore apply Theorem 3.2 to this case, but we first have to ensure that the functions ϕ_C and ψ_C satisfy the assumptions of that theorem. This will be the case if we assume that each $C \in \mathcal{C}$ has *constant description complexity*, that is, it is defined by a constant number of algebraic equalities and inequalities of constant maximum degree. In this case one can easily show that the functions ϕ_C and ψ_C do indeed satisfy the required conditions.[3] (The general position assumption can be enforced by imposing a similar general position assumption on the sets in $\mathcal{C}$.) We thus have:

Theorem 3.4 *The complexity of the space of plane transversals of a collection of n compact convex sets in 3-space, each of constant description complexity, is $O(n^{2+\varepsilon})$, for any $\varepsilon > 0$.*

Remarks: (1) Convexity is not essential here, since we can replace each set in $\mathcal{C}$ by its convex hull, without affecting the transversality of any plane.
(2) If the sets in $\mathcal{C}$ are not each of constant description complexity, the complexity of $T(\mathcal{C})$ can be arbitrarily large. However, if one assumes, in addition, that the sets are *separated*, in the sense that no three of these sets is crossed by a common line, then it is shown by Cappell et al. [4] that, for such a collection $\mathcal{C}$, the complexity of $T(\mathcal{C})$ is $O(n^2)$. This bound, in this restricted case, is slightly better than the bound derived above. The result of [4] applies in higher dimensions too: Under an appropriate assumption of separation of the sets in $\mathcal{C}$, the complexity of $T(\mathcal{C})$ is $O(n^{d-1})$.

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