# The Visibility Region of Points in a Simple Polygon*

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## Abstract

Let $R$ be a polygonal region with $h$ polygonal holes and $n$ vertices in total, and let $P$ be a set of $m$ point guards in the interior of $R$. We show that the region of all points in $R$ visible from at least one guard in $P$ has at most $n+2mn+4\binom{h+2}{2}\binom{m}{2}$ vertices and can be computed in time $O(((m(h+1))^2+mn\log m)\log(m+n))$.

## 1 Introduction

Visibility problems have a long history in computational geometry. Perhaps the most fundamental of these are the so-called art-gallery problems. In its simplest form, an art gallery is a simple polygon, perhaps with holes, and we ask how many guards (points) inside the polygon are necessary to guard the whole polygon. Many variations on this theme have been studied, and a whole book surveys results on art-gallery problems [2].

We consider the situation where the guards are already given, and we are interested in computing the guarded region. To stay within the art-gallery metaphor, we have placed the guards, and we need to determine the safe parts of the gallery—where artifacts can be placed under the supervision of the guards.

Surprisingly, except for the case of a single guard, this problem seems not to have been studied before. We show that the number of vertices of the visibility region of $m$ points in the interior of a polygonal region $R$ with $h$ polygonal holes and $n$ vertices in total is at most $n+2mn+4\binom{h+2}{2}\binom{m}{2}$, i.e., the term quadratic in $m$ does not depend on $n$. We also give lower bound examples, and an algorithm for computing the visibility region in $O(((m(h+1))^2+mn\log m)\log(m+n))$ time.

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## 2 Preliminaries

Given a simple polygon $T$ in the plane, let $\partial T$ denote the boundary of $T$, and let $Int(T)$ denote the interior of $T$. Let now $R_0,\dots R_h$ be simple polygons such that $R_i \subset Int(R_0)$ for each $1 \leq i \leq h$, and $R_i \cap R_j = \emptyset$ for each $1 \leq i < j \leq h$. Let $R$ be $R_0 \setminus \cup_{1\leq i\leq h} Int(R_i)$. Informally, $R$ is a polygonal region with holes $R_1,\dots,R_h$. The boundary $\partial R$ of $R$ is $\bigcup_{0\leq i\leq h}\partial R_i$ and consists of $h+1$ components; the interior $Int(R)$ is $Int(R_0)\setminus \cup_{1\leq i\leq h} R_i$. Let $n$ be the number of vertices of $R$.

Let $P$ be a set of $m$ points in $R$ (that is, in its interior or on the boundary). A point $p \in P$ and a point $q$ in $R$ (but not necessarily in $P$) are called mutually visible if the segment $pq$ does not intersect the exterior of $R$. The *visibility region* $V(p)$ of a point $p \in P$ is defined as the locus of all points $q \in R$ that are visible from $p$. Let $V(P)$ be $\bigcup_{p\in P} V(p)$. We are interested in the combinatorial complexity of $V(P)$, that is the number of vertices on the boundary of $V(P)$.

A vertex $v$ of $R$ is called a *reflex vertex* if a line segment $s$ exists that contains $v$ in its interior but is itself fully contained in $R$. Let $r$ be the number of reflex vertices of $R$.

For a point $p \in P$ and a reflex vertex $v$ of $R$, let $s(p,v)$ be the segment generated as follows: the ray with origin $v$ and direction $\vec{pv}$ intersects $\partial R$ in zero or more points. For at most one $v'$ of these intersection points, the interior of the segment $vv'$ lies completely in $Int(R)$. If there is such a point, $s(p,v)$ is defined as the segment $vv'$. Otherwise, $s(p,v)$ is undefined. See also Figure 1, and note that this definition of $s(p,v)$ also covers degenerate situations, such as collinearity of a point $p \in P$ and multiple vertices of $R$. All segments $s(p,v)$ generated by $p$ and a vertex $v$ of $R$ lie on the boundary of $V(p)$. For a point $p \in P$ and a reflex vertex $v$ of $R_i$, we call $s(p,v)$ *left-bounded* if $R_i$ lies locally to the left of the ray from $p$ through $v$, and *right-bounded* otherwise. Note that one of the endpoints of $s(p,v)$ is the vertex $v$ itself, the other endpoint lies on $\partial R$.

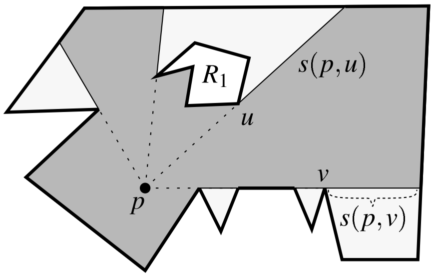


Figure 1: Segments $s(p,v)$ on the boundary of $V(p)$.

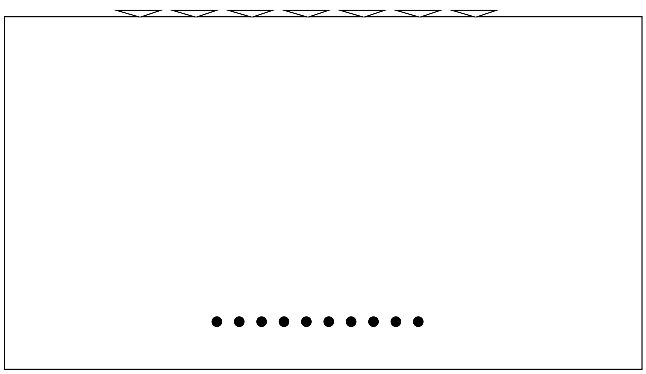

Figure 2: Lower bound example for vertices on the boundary

## 3 Complexity of the visibility region

For a single point $p \in P$, the vertices of $V(p)$ are either vertices of $R$ or endpoints of segments $s(p,v)$ for some vertex $v$ of $R$. Since there are at most $r$ such segments for a single point $p$, and since each segment has at most one endpoint that is not a vertex of $R$, the number of vertices of $V(p)$ is at most $n+r$. In fact, a slightly more careful analysis shows the number of vertices is at most $n$.

A vertex of $V(P)$ is either a vertex of $V(p)$ for some $p \in P$, or is the intersection point of two segments $s(p,v)$ and $s(q,w)$ for $p,q \in P$ and $v,w$ reflex vertices of $R$. It follows that the number of vertices on the boundary of $V(P)$ is at most $n+mr+\binom{m}{2}\binom{n}{2}$. Our main result shows that this naive bound can be improved to $n+2mr+4\binom{m}{2}\binom{h+2}{2}$. In other words, the term quadratic in $m$ does not depend on $n$ at all.

We will prove this bound by looking at different categories of vertices of $V(P)$. We start with the simple bound on the number of vertices on the boundary.

**Lemma 1** *The number of vertices of $V(P)$ on the boundary of $R$ is at most $n+rm$.*

**Proof:** Every such vertex is either a vertex of $R$, or the endpoint of a segment $s(p,v)$. There are at most $rm$ vertices of this second kind.

Figure 2 gives an example of a region where the number of vertices on the boundary is $n-r+rm$. ◻

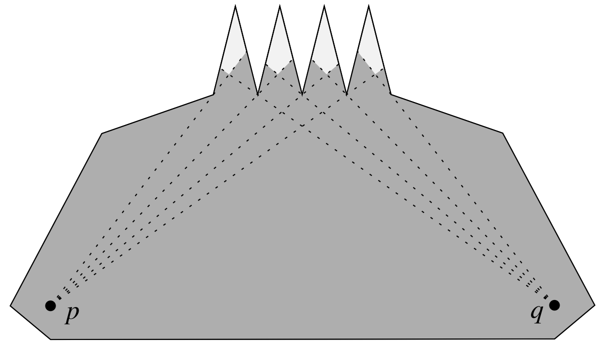


Figure 3: Two points generate $r-1$ interior vertices

It remains to bound the number of vertices of $V(P)$ in the interior of $R$—we will call them *interior vertices*. There are no interior vertices if $P$ is a single point. If $P$ consists of two or more points, then we divide the interior vertices into two groups, the *reachable vertices* and the *unreachable vertices*. We call an interior vertex $w$ formed by $s(p,v)$ and $s(q,u)$ *reachable* if one of the triangles $pwu$ or $qwv$ lies in $R$. The interior vertices in Figure 3 are all reachable vertices. An interior vertex is called *unreachable* if it is not reachable. We show that the number of reachable vertices is at most $rm$. If $P$ consists of two points $p$ and $q$, then the number of unreachable vertices is "basically" independent of $n$ and bounded by $4\binom{h+2}{2}$. This turns out to be enough to prove a good bound for general point sets $P$: Since every interior vertex of $V(P)$ is defined by two points $p,q \in P$, it must appear as an interior vertex in $V(\{p,q\})$, and so the total number of interior vertices can be bounded by $rm+4\binom{h+2}{2}\binom{m}{2}$.

**Lemma 2** *The number of reachable vertices is bounded by $rm$, and is at least $r-1$ in the worst case.*

**Proof:** Figure 3 proves the lower bound.

We will prove the upper bound by charging each reachable vertex $w$ to a unique visibility segment $s(p,v)$. In fact, if $w$ is defined by $s(p,v)$ and $s(q,u)$, and the triangle $qwv$ lies in $R$, we charge $w$ to $s(p,v)$.

It remains to see that every visibility segment can be charged at most once. This follows from the fact that $w$ must be the first vertex of $V(P)$ on $s(p,v)$. ◻

To bound the number of unreachable vertices, we can restrict our attention to the visibility region of two points $p$ and $q$. We will call an unreachable vertex $w$ of $V(\{p,q\})$ formed as the intersection of $s(p,v)$ and $s(q,u)$ an *entry vertex* if either $w$ lies to the right of the directed line from $p$ to $q$, $s(p,v)$ is left-bounded and $s(q,u)$ is right-bounded, or $w$ lies to the left of the directed line from $p$ to $q$, $s(p,v)$ is right-bounded and $s(q,u)$ is left-bounded. See Figure 4.

**Lemma 3** *The number of unreachable vertices of $V(\{p,q\})$ is at most four times the number of entry vertices.*

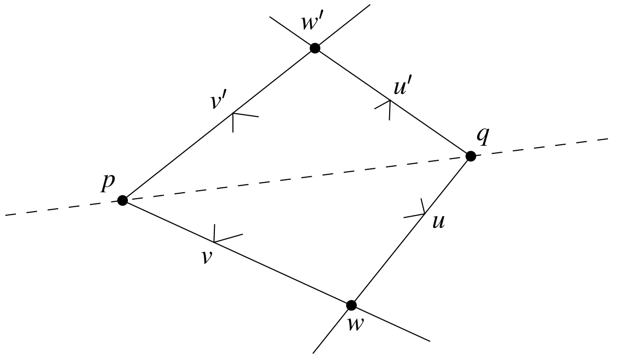


Figure 4: $w$ and $w'$ are entry vertices

**Proof:** We give the proof for the vertices lying to the right of the directed line through $p$ and $q$, the other case being symmetrical.

Consider a visibility segment $s(p,v)$. It may contain several unreachable vertices of $V(\{p,q\})$. We observe that these vertices are formed by segments $s(q,u)$ that are alternatingly right-bounded and left-bounded, starting with a left-bounded one. Similarly, the unreachable vertices on a visibility segment $s(q,u)$ are alternatingly right-bounded and left-bounded, starting with a left-bounded one.

We give a charge of 4 to every entry vertex of $V(\{p,q\})$. Consider now each left-bounded visibility segment $s(p,v)$. Its unreachable vertices are alternatingly right-bounded and left-bounded. Since the right-bounded ones are entry vertices, we can distribute their charge to the remaining unreachable vertices. After this step, every unreachable vertex on all left-bounded segments $s(p,v)$ has a charge of 2.

Consider now each segment $s(q,u)$. Its unreachable vertices are alternatingly left-bounded and right-bounded, starting with a left-bounded one. Since the left-bounded ones have a charge of 2, we can thus distribute the charge to all unreachable vertices on the segment and end up giving each unreachable vertex a charge of 1. $\square$

The following lemma allows us to give the final upper bound.

**Lemma 4** *Given a polygonal region $R$ with $h$ holes and two points $p$ and $q$ in $R$. Then the number of unreachable vertices of $V(\{p,q\})$ is at most $4\binom{h+2}{2}$, and at least $4\binom{h+1}{2}$ in the worst case.*

**Proof:** The lower bound construction is given in Figure 5.

By Lemma 3 it is sufficient to prove that the number of entry vertices is at most $\binom{h+2}{2}$. For simplicity of presentation, assume that $p$ and $q$ lie on a horizontal line $\ell$.

Let's first assume that all entry vertices lie below $\ell$. An entry vertex $w$ is defined by two segments $s(p,v)$ and $s(q,u)$, where $v$ and $u$ are vertices of two components $R_i$ and $R_j$ of $R$, where $0 \le i,j \le h$. We argue that there is at most one entry vertex for every choice of $R_i$ and $R_j$. Assume to the contrary that there are two different entry vertices $w,w'$

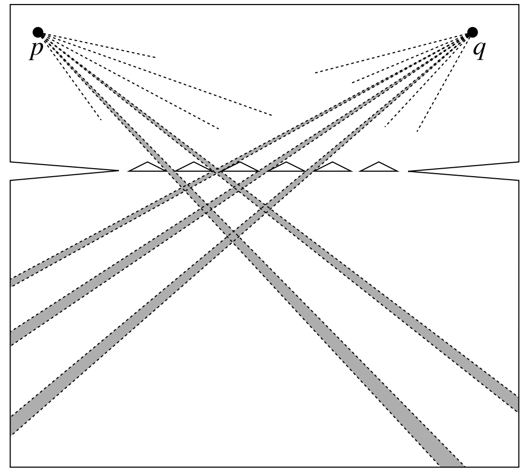


Figure 5: An example with $4\binom{h+1}{2}$ unreachable vertices generated by two points.

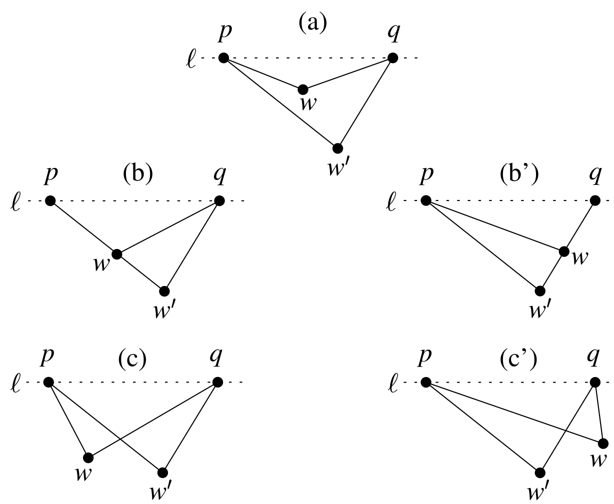


Figure 6: The five cases.

defined by $R_i$ and $R_j$. Let $w'$ be the vertex closer to $\ell$. There are five different configurations, see Figure 6. Clearly, in cases (a), (b), and (c) it is impossible for a component $R_i$ to touch $qw$ and either $pw'$ or $qw'$ from above. Cases (b') and (c') are excluded by a symmetric argument.

Thus for every choice of $R_i$ and $R_j$ there is at most one entry vertex. Since there are $h+1$ choices where $R_i = R_j$, and $\binom{h+1}{2}$ choices where $R_i \neq R_j$, it follows that the number of entry vertices is at most $\binom{h+2}{2}$.

We now consider the case that there are entry vertices above and below $\ell$. Let $h_1$ and $h_2$ be the number of holes lying completely above resp. below $\ell$, and let $h_3$ be the number of holes intersecting $\ell$.

Let's first assume that $h_3 = 0$, so all holes lie completely to one side of $\ell$. Note that in this situation the outer boundary $\partial R_0$ cannot participate in forming an entry vertex, and so the number of entry vertices can be bounded by $\binom{h_1+1}{2}$

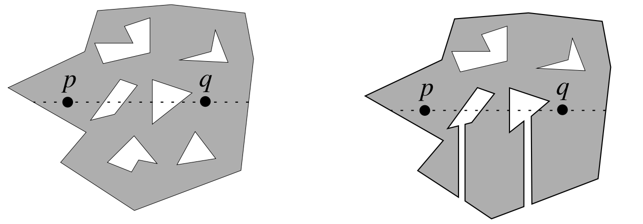


Figure 7: Holes intersecting $\ell$ can be eliminated

and $\binom{h_2+1}{2}$. Since $h_1+h_2=h$, we have $\binom{h_1+1}{2}+\binom{h_2+1}{2}=\binom{h+2}{2}-(h_1+1)(h_2+1)\leq\binom{h+2}{2}$.

It remains to consider the possibility that $h_3>0$. Again we count the entry vertices above and below $\ell$ separately. While counting the entry vertices above $\ell$, we can discard all holes below $\ell$, and we can connect the holes intersecting $\ell$ to $\partial R_0$ as in Figure 7. The number of entry vertices above $\ell$ can thus be bounded by $\binom{h_1+2}{2}$, and in the same way we can bound the number of entry vertices below $\ell$ by $\binom{h_2+2}{2}$. Since $h_1+h_2=h-h_3<h$, we have

$$\begin{aligned}
& \binom{h_1+2}{2}+\binom{h_2+2}{2} \\
= \; & (h_1+1)+\binom{h_1+1}{2}+\binom{h_2+2}{2} \\
= \; & (h_1+1)+\binom{h_1+h_2+3}{2}-(h_1+1)(h_2+2) \\
\leq \; & \binom{h+2}{2}.
\end{aligned}$$

◻

**Theorem 1** *Let $R$ be a polygonal region with $h$ polygonal holes, $r$ reflex vertices and $n$ vertices in total, and let $P$ be a set of $m$ points in $R$. Then the visibility region $V(P)$ has at most $n+2rm+4\binom{h+2}{2}\binom{m}{2}$ vertices.*

*For any value of $r$ and $m$ and $n$ sufficiently large, there are regions $R$ and sets $P$ such that $V(P)$ has at least $(n-2(h+1))+rm+h(m-1)+4\binom{h+1}{2}\binom{m}{2}$ vertices.*

**Proof:** By Lemmas 1 and 2, the number of reachable and boundary vertices is at most $n+2rm$.

Every unreachable vertex $w$ of $V(P)$ is defined by two points $p,q\in P$, and so $w$ must appear as an unreachable vertex in $V(\{p,q\})$. Since there are $\binom{m}{2}$ pairs of points $p$ and $q$, and by Lemma 4, the total number of unreachable vertices is at most $4\binom{h+2}{2}\binom{m}{2}$.

An example proving the lower bound is given in Figure 8. See the full paper for details. ◻

A gap remains in the constant factors between the upper and the lower bound.

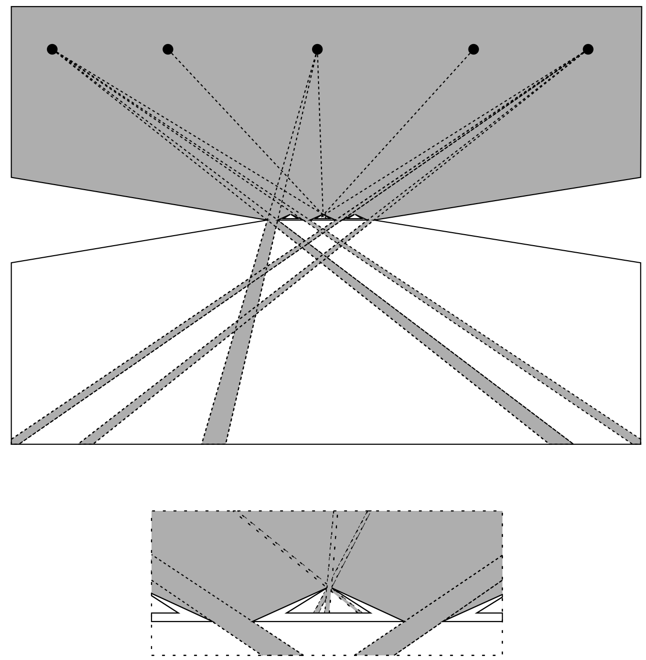

Figure 8: Lower bound example, with detail of hole (below).

## 4 Algorithms

The visibility region $V(P)$ can be constructed using a divide and conquer approach. We divide $P$ into two sets $P',P''$ of equal size, recursively compute $V(P')$ and $V(P'')$, and merge the regions into $V(P)$ using a standard plane sweep. in $O(n)$ time for a simple polygon [1, 2, pp. 203–206], and the visibility region of a single point $p\in P$ can be constructed in $O(n\log n)$ time for a polygon with holes [2, pp. 217–219].

In the case of a polygon with $h$ holes, the running time $T(m)$ for constructing the visibility region for $m$ points in $P$ therefore satisfies the following recurrence.

$$\begin{aligned}
T(1) &= O(n\log n) \\
T(m) &= 2T(m/2)+O(((h+1)m)^2+mn)\log(m+n))
\end{aligned}$$

This solves to $T(m)=O(((m(h+1))^2+mn\log m)\log(m+n))$. We have the final theorem:

**Theorem 2** *Given a polygonal region $R$ with $h$ holes and $n$ vertices in total, and a set $P$ of $m$ points in $R$, the visibility region $V(P)$ can be computed in time $O(((m(h+1))^2+mn\log m)\log(m+n))$.*

## References

[1] B. Joe and R. B. Simpson. Correction to Lee's visibility polygon algorithm. *BIT*, 27:458–473, 1987.

[2] J. O'Rourke. *Art Gallery Theorems and Algorithms*. The International Series of Monographs on Computer Science. Oxford University Press, New York, NY, 1987.